AN EXPLICIT TRACE FORMULA OF JACQUET-ZAGIER TYPE AND ITS APPLICATIONS

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1. Introduction

Let $\mathbf{H} = \{\tau = x + iy |\, x \in \mathbb{R},\, y > 0\}$ be the Poincaré upper half space with the hyperbolic metric $\mathrm{d}s^2 = (\mathrm{d}x^2 + \mathrm{d}y^2)/y^2$. The associated volume form is $\mathrm{d}\mu(\tau) = \frac{\mathrm{d}x\mathrm{d}y}{y^2}$ which is $\mathrm{SL}_2(\mathbb{R})$-invariant. Let $k \in 2\mathbb{N}$ be a positive even integer. For any $N \in \mathbb{N}$, let $S_k(\Gamma_0(N))$ denote the space of all the elliptic cusp forms on $\Gamma_0(N)$ with weight k; as usual, the space carries the Petersson inner product defined by

$$\langle f, f_1\rangle = \int_{\Gamma_0(N)\backslash \mathbf{H}} f(\tau)\bar{f}_1(\tau)(\mathrm{Im}\tau)^k \mathrm{d}\mu(\tau), \quad f, f_1 \in S_k(\Gamma_0(N)).$$

Let $H_k(N)$ be an orthonormal basis of $S_k(\Gamma_0(N))$ consisting of simultaneous eigenfunctions of Hecke operators $T(n)$ with $n \in \mathbb{N}$ relatively prime to the level N; the normalized eigenvalue of $T(n)$ on $h \in H_k(N)$ is denoted by $\lambda_h(n)$, i.e.,

$$T(n)h = n^{(k-1)/2}\lambda_h(n)\, h, \quad h \in H_k(N),\ (n, N) = 1.$$

Let ϕ be an even Hecke-Maass form on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbf{H}$, i.e., $\phi(\tau)$ is a C^∞-fucntion belonging to $L^2(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbf{H})$ with $\int_0^1 \phi(x+iy)\,\mathrm{d}x = 0\ (\forall y > 0)$ satisfying the joint eigen-equations

$$\Delta\phi(\tau) := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2}\right)\phi(\tau) = \lambda_\infty\, \phi(\tau), \tag{1.1}$$

$$T(p)\phi(\tau) := \phi(p\tau) + \sum_{j=0}^{p-1}\phi\left(\tfrac{\tau+j}{p}\right) = p^{1/2}\,\lambda_p\,\phi(\tau) \quad (p:\ \text{prime number}). \tag{1.2}$$

and possessing the invariance under the orientation reversing automorphism $\tau \to -\bar{\tau}$ of $\mathbf{H}$. We always assume that ϕ is normalized so that its first Fourier coefficient is 1, i.e.,

$$\phi(x+iy) = \sum_{n\in\mathbb{Z}-\{0\}} \lambda_\phi(n)\, y^{1/2} K_{\nu_\infty/2}(2\pi|n|y)\, e^{2\pi i n x}$$

with $\lambda_\phi(1) = 1$, where $\nu_\infty \in i\mathbb{R}$ is determined by the relation $\lambda_\infty = (1-\nu_\infty^2)/4$. [1] Note that $\lambda_\phi(p) = \lambda_p$ for all primes p. Then the trace formula of Jacquet-Zagier type is a formula which

[1]For $\mathrm{SL}_2(\mathbb{Z})$, the estimate $\lambda_\infty \geqslant 1/4$ (i.e., $\nu_\infty \in i\mathbb{R}$) is known.

gives us an exact evaluation of the spectral average

$$\sum_{h\in H_k(N)} \underbrace{\lambda_h(n)}_{\boxed{\text{Hecke eigenvalue}}} \times \underbrace{\int_{\Gamma_0(N)\mathbf{H}} \phi(\tau)\, h(\tau)\, \bar{h(\tau)}(\mathrm{Im}\tau)^k \mathrm{d}\mu(\tau)}_{\boxed{\text{triple product period}}}. \tag{1.3}$$

When $\phi = 1$, the exact evaluation of this average had been well-known as the *Eichler-Selberg trace formula* of the Hecke operator $T(n)$. When ϕ is the non-holomorphic Eisenstein series

$$E(z,\tau) = \sum_{\gamma\in\{\pm\left[\begin{smallmatrix}1 & *\\ 0 & 1\end{smallmatrix}\right]\}\backslash \mathrm{SL}_2(\mathbb{Z})} \mathrm{Im}(\gamma\tau)^{(z+1)/2}, \quad \mathrm{Re}\, z > 1,\, \tau \in \mathbf{H}$$

on $\mathrm{SL}_2(\mathbb{Z})$ and when $N = 1$, Zagier obtained an explicit formula of the average in terms of the L-series of the quadratic fields and observed that the Eichler-Selberg trace formula can be reproved by taking the residue at $z = 1$ ([18]). Later, the method introduced by Zagier is carried over to the case of the Hilbert modular forms over a totally real number field of narrow ideal class number 1 by Mizumoto ([6]) and Takase ([17]). In [4], Jacquet and Zagier considered the problem in a most general and abstract framework on the adelization of GL(2) over an arbitrary number field. In this short report, we explain our recent progress on a new method to prove the trace formula of Jacquet-Zagier type in a bit different way than those in these works, which allows us to settle the case when ϕ is a cusp form at the same time.

In the context of quantum chaos on the modular surface $Y_0(N) = \Gamma_0(N)\backslash\mathbf{H}$, the average (1.3) is also meaningful, where the triple product occurring in (1.3) is vewed as a probability measure on $Y_0(N)$ depending on $h \in H_k(N)$, i.e.,

$$\mu_h(\psi) = \int_{\Gamma_0(N)\backslash\mathbf{H}} \psi(\tau)\, |h(\tau)|^2\, (\mathrm{Im}\tau)^k \mathrm{d}\mu(\tau)$$

for any bounded measurable function ψ on the surface $Y_0(N)$. The holomorphic analogue of Quantum Unique Ergodicity conjecture originary proposed by Rudnick and Sarnak ([7]) asserts that the measure μ_h for every $h \in H_k(N)$ should converge to the invariant probability measure $\mathrm{vol}(\Gamma_0(N)\backslash\mathbf{H})^{-1}\mu$ on $Y_0(N)$ ("the semiclassical limit") as the weight k grows to infinity. Before the holomorphic QUE conjecture, as well as the original QUE conjecture itself, has been proved ([9], [12]) for the full modular case $N = 1$, several research concerning the average of the family μ_h $(h \in H_k(1))$ are conducted[2]. For example, Luo [13] showed the holomorphic QUE conjecture on average proving the limit formula

$$\frac{1}{\#H_k(1)} \sum_{h\in H_k(1)} \mu_h(\psi) \quad\to\quad \mathrm{vol}(\Gamma_0(1)\backslash\mathbf{H})^{-1}\mu(\psi), \quad \psi \in C_0^\infty(Y_0(1)) \tag{1.4}$$

for the full level case[3]. The limiting behavior as $\mathrm{K} \to +\infty$ of the "quantum variance"

$$\sum_{k\in 2\mathbb{N}} u\left(\frac{k-1}{\mathrm{K}}\right) \sum_{h\in H_k(1)} |\mu_h(\psi)|^2 \tag{1.5}$$

and the modified version

$$\sum_{k\in 2\mathbb{N}} u\left(\frac{k-1}{\mathrm{K}}\right) \sum_{h\in H_k(1)} L(1,h,\mathrm{sym}^2)\, |\mu_h(\psi)|^2, \tag{1.6}$$

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[2]The relationship between our results and the QUE conjecture on average should have been addressed properly in the talk. Here we put a follow-up information for the people in the audience.

[3]As a matter of fact, in [13] this formula is shown only when ψ is the characteristic function of a bounded measurable set of $Y_0(1)$. The proof works for general ψ.

are considered by Luo-Sarnak ([14]), where u is an arbitrary function from $C_0^\infty(0,+\infty)$ and $L(s,h,\mathrm{sym}^2)$ is the symmetric square L-series of h. In the above formulas, the "test function" ψ is supposed to be taken from the space $C_0^\infty(Y_0(1))$ (resp. $C_{0,0}^\infty(Y_0(1))$) of all the smooth functions (resp. those with zero means $\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbf{H}}\psi(\tau)\,\mathrm{d}\mu(\tau)=0$) which as well as their derivatives satisfy the bound $|\psi(\tau)|\ll_A y^{-A}$ on $y>\sqrt{3}/2$ for any $A>0$. They showed that (a) there exists a certain Hermitian form $B_\omega(\psi)$ on the space $C_{0,0}^\infty(Y_0(1))$ such that (1.6) is asymptotically equal to

$$B_\omega(\psi)\left(\int_0^\infty u(t)\mathrm{d}t\right)\mathrm{K}+O_{\epsilon,\psi}(\mathrm{K}^{1/2+\epsilon})$$

and (b) the quantity $B_\omega(\phi)$ for an even Hecke-Maass cusp form ϕ as above is given as $\frac{\pi}{2}\,L(\frac{1}{2},\phi)\langle\phi,\phi\rangle$. They gave a brief remark indicating a similar asymptotic formula for the average (1.5) should hold true. However, the necessary argument is not that direct, after a while, some detail concerning to the proof appeared in the preprint [15]. Stricltly speaking [15] consideres the variance for the family μ_{ϕ_j} over an orthonormal system of Hecke-Maass forms $\{\phi_j\}$ in the realm of the original QUE conjecture; we can check that the argument is easily carried over to the holomorphic case. If we choose u to be an appropriate approximation of the characteristic function of the interval $[1,2]$, then the arguments end up with the asymptotic formula (*cf* [15, Corollary 1]) :

$$\sum_{\substack{k\in 2\mathbb{N}\\ \mathrm{K}\leqslant k<2\mathrm{K}}}\ \sum_{h\in H_k(1)}|\mu_h(\phi)|^2\sim\frac{\pi\,\langle\phi,\phi\rangle}{2}L\left(\tfrac{1}{2},\phi\right)\,C(\phi)\,\mathrm{K}\quad(\mathrm{K}\to+\infty). \tag{1.7}$$

The quantity $C(\phi)$ is given by the following convergent Euler poduct over prime numbers

$$C(\phi):=\frac{1}{\zeta(2)}\prod_p\left(1-\frac{p^{-1}\lambda_p}{p^{1/2}+p^{-1/2}}\right),$$

where λ_p is the p-th normalized Hecke eigen value of ϕ fitting in the eigenequation (1.2), by which the Hecke L-function appering in the asymptotic formula is defined as the analytic continuations of the degree 2 Euler product

$$L(s,\phi)=\prod_p(1-\lambda_p\,p^{-s}+p^{-2s})^{-1},\quad \mathrm{Re}\ s>1.$$

We remark that the completed L-function

$$\Lambda(s,\phi)=\Gamma_{\mathbb{R}}(s+\nu_\infty/2)\Gamma_{\mathbb{R}}(s-\nu_\infty/2)\,L(s,\phi)$$

with $\nu_\infty\in i\mathbb{R}$ determined from the Laplace eigenvalue λ_∞ by $\lambda_\infty=(1-\nu_\infty^2)/4$ is entire and satisfies the functional equation $\Lambda(s,\phi)=\Lambda(1-s,\phi)$ with plus sign.

Our main results to be reported in this write-up is summarized as follows

(i) Restricting ourselves to the full modular case (i.e., $N=1$), we describe an exact formula of (1.3) when ϕ is the normalzied Eisenstein series $E^*(z)$ or a cusp form in a uniform way.

(ii) We state an asymptotic formula of the average $\sum_{h\in H_k(1)}\mu_h(\phi)\,\lambda_h(n)$ for any fixed $n\in\mathbb{N}$ with the weight k growing to infinity. Luo's result (1.4) applied to an even Hecke-Maass form ϕ yields$(\#H_k(1))^{-1}\sum_{h\in H_k(1)}\mu_h(\phi)\to 0$ as $k\to\infty$ because $\mu(\phi)=0$ by the cuspidality. Our asymptotic formula (Theorem 3.2) improves this result in that it tells us a rate of convergence to 0.

(iii) From our asymptotic formula in cojunction with (1.7), we deduce a quantitative non-vanishing of the central L-values for the family of degree 6 L-functions $L(s, \phi \times \mathrm{Ad}(h))$ $(h \in \cup_{k \in [\mathrm{K}, 2\mathrm{K})} H_k(1))$ with growing K.

(iv) A weighted version of the vertical Sato-Tate law which is viewed as a deformation of Serre's result [8] in an accurate sence. The ensamble is constructed by the special values on a point in the critical strip of the symmetric square L-functions $L(s, h; \mathrm{sym}^2)$ $(h \in H_k(N))$ with a fixed weight $k \geqslant 4$ with growing square free levels N.

Remark : In [10], we work with a Hecke normalized basis rather than an orthonormal basis. Let $\mathcal{B}_k(N)$ be the set of all the Hecke eigen forms in $S_k(\Gamma_0(N))$ under the Hecke operators $T(n)$ with $(n, N) = 1$ such that the first Fourier coefficient is normalized to be 1. Then we can take $H_k(N) = \{f \langle f, f\rangle^{-1/2} |\, f \in \mathcal{B}_k(N)\}$.

2. Period formulas

As usual we set $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$. Let ϕ and $h \in H_k(1)$ be as above. Recall the symmetric square L-function of ϕ and its completion are defined as

$$L(s, \phi; \mathrm{sym}^2) = \prod_p (1 - p^{-\nu_p - s})^{-1}(1 - p^{-s})^{-1}(1 - p^{\nu_p - s})^{-1}, \quad \mathrm{Re}\, s > 1,$$

$$\Lambda(s, \phi; \mathrm{sym}^2) = \Gamma_{\mathbb{R}}(s + \nu_\infty)\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s - \nu_\infty) \times L(s, \phi, \mathrm{sym}^2),$$

where $\pm\nu_\infty$ and $\pm\nu_p$ are the spectral parameters of ϕ defined by the relations

$$\left(\tfrac{1}{2} - \tfrac{\nu_\infty}{2}\right)\left(\tfrac{1}{2} + \tfrac{\nu_\infty}{2}\right) = \lambda_\infty, \quad \lambda_p = p^{-\nu_p/2} + p^{\nu_p/2} \quad (p: \text{prime numbers}),$$

respectively. For $h \in H_k(1)$, $L(s, h; \mathrm{sym}^2)$ and its completion are defined similarly. For convenience, we record several well-known period formulas for our automorphic forms.

$\boxed{\textit{Petersson inner products}}$:

The Rankin-Selberg method yields the identities:

$$\langle \phi, \phi \rangle = \tfrac{1}{2}\Lambda(1, \phi, \mathrm{sym}^2), \qquad \langle h, h \rangle = (4\pi)^{-k}\pi^{-1}\Gamma(k)\, L(1, h.\mathrm{sym}^2), \quad h \in H_k(1).$$

$\boxed{\textit{Waldspurger's period formula}}$:

Let $\mathbb{P}_D(\phi)$ with $D \in \mathscr{D}$ is the period integral of ϕ to be recalled in § 3.1. From [5, Theorem 4.1], we obtain

$$\frac{|\mathbb{P}_D(\phi)|^2}{\|\phi\|^2} = \frac{\sqrt{|D|}}{4}\frac{L(1/2, \phi)\, L(1/2, \phi \otimes \chi_D)}{L(1, \phi, \mathrm{sym}^2)}.$$

for any $D \in \mathscr{D}$, $D < 0$. The gamma factor of $L(\sigma, \phi \otimes \chi_D)$ is given as $\Gamma_{\mathbb{R}}(s + \nu_\infty/2 + 1)\Gamma_{\mathbb{R}}(s - \nu_\infty/2 + 1)$ if $D < 0$.

$\boxed{\textit{Watson-Ichino formula}}$:

We quote Watson's formula from [14, p.785 last line] ([20])[4]. This is a special case of Ichino's formula [3].

$$|\mu_h(\phi)|^2 = \frac{\pi^2}{2\cos(\pi\nu_\infty/2)}\frac{|\Gamma(k - \frac{1}{2} + \frac{\nu_\infty}{2})|^2}{(4\pi)^k\Gamma(k)}\, |a_h(1)|^2\, \frac{L\left(\frac{1}{2}, \phi \otimes h \otimes h\right)}{L(1, h, \mathrm{sym}^2)}, \quad h \in H_k(1),$$

[4]Note that our $\pm\nu_\infty$ is $\pm it_\phi/2$ in [14].

where $h=\sum_{n=1}^{\infty} a_h(n)q^n$ is the q-expansion of h. Recall the decomposition formula of the triple product L-function:

$$L(s,\phi\otimes h\otimes h)=L(s,\phi)\,L(s,\phi\times \mathrm{Ad}\,h).$$

where $L(s,\phi\otimes \mathrm{Ad}\,h)$ denotes a degree 6 L-function defined as the $\mathrm{GL}(2)\times\mathrm{GL}(3)$ convolution L-function of π_ϕ and the Gelbart-Jacquet lift $\mathrm{Ad}(\pi_h)$ ([2]) with π_ϕ and π_h being the automorphic representations of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ generated by ϕ and h, respectively.

3. Results

3.1. **Trace formula.** Let $\mathscr{D}$ be the set of all the fundamental discriminants. For each $D\in\mathscr{D}$, the Kronecker character of the quadratic field $\mathbb{Q}(\sqrt{D})$ is denoted by χ_D. Since D is fundamental, χ_D is a quadratic Dirichlet character of conductor D. For $D\in\mathscr{D}$, let $\mathcal{F}(D)$ be the set of all the integral binary quadaratic forms $Q(x,y)=Ax^2+Bxy+Cy^2\in\mathbb{Z}[x,y]$ such that $\gcd(A,B,C)=1$, $B^2-4AC=D$, and $Q(x,y)$ is not negative definite. Then the group $\mathrm{PSL}_2(\mathbb{Z})$ acts on the set $\mathcal{F}(D)$ by the rule

$$(Q\cdot\gamma)(x,y)=Q(ax+by,cx+dy),\quad \text{for } \gamma=\left[\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right]\in\mathrm{SL}_2(\mathbb{Z}).$$

Let us recall a few facts on the orbit space $\mathcal{F}(D)/\mathrm{PSL}_2(\mathbb{Z})$. First of all, the orbits are finite in number $h=\#(\mathcal{F}(D)/\mathrm{PSL}_2(\mathbb{Z}))$, which equals the narrow class number of the quadratic field $\mathbb{Q}(\sqrt{D})$. For $Q\in\mathcal{F}(D)$, the stabilizer subgroup of Q is

$$\Gamma(Q)=\{\gamma\in\mathrm{PSL}_2(\mathbb{Z})|\,Q\cdot\gamma=Q\}\cong U_D/\{\pm1\},$$

where U_D is the global units of $\mathbb{Q}(\sqrt{D})$.

$\boxed{\text{Period integrals}}$:
The "period integral" of ϕ for $D\in\mathscr{D}$ to be denoted by $\mathbb{P}_D(\phi)$ is defined as follows. If $D<0$,

$$\mathbb{P}_D(\phi):=\frac{2}{\#U_D}\sum_{Q\in\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{F}(D)}\phi(z_Q),$$

where z_Q is the unique root of $Q(z,1)=0$ with $\mathrm{Im}(z_Q)>0$.
If $D>0$,

$$\mathbb{P}_D(\phi):=\sum_{Q\in\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{F}(D)}\int_{\Gamma(Q)\backslash C_Q}\phi(\tau)\,|\mathrm{d}_Q\tau|,$$

where C_Q is the geodesic line (semi-circle) on $\mathbf{H}$ joining two real roots of $Q(z,1)=0$ and $|\mathrm{d}_Q s|$ the geodesic line element on C_Q.
When $\Delta=f^2D$ with $f\in\mathbb{N}$ and $D\in\mathscr{D}$, we set $\mathbb{P}_\Delta(\phi)=\mathbb{P}_D(\phi)$.

$\boxed{p\text{-adic factors}}$:
For any $\Delta\in\mathbb{Z}-\{0\}$ such that $\Delta\equiv 0,1 \pmod 4$, we have the expression $\Delta=Df^2$ (uniquely) with $f\in\mathbb{N}$ and $D\in\mathscr{D}\cup\{1\}$. For any $\nu=\{\nu_p\}_{p<\infty}\in\prod_p(\mathbb{C}/4\pi i(\log p)^{-1}\mathbb{Z})$, we set

$$\mathbf{B}(\nu;\Delta)=\prod_{p|f}\left\{\frac{\zeta_p(-\nu_p)}{L_p\left(\frac{-\nu_p+1}{2},\chi_D\right)}|f|_p^{\frac{\nu_p-1}{2}}+\frac{\zeta_p(\nu_p)}{L_p\left(\frac{\nu_p+1}{2},\chi_D\right)}|f|_p^{\frac{-\nu_p-1}{2}}\right\},$$

where $\zeta_p(s)=(1-p^{-s})^{-1}$, $L_p(s,\chi_D)=(1-\chi_D(p)p^{-s})^{-1}$, and $|f|_p$ the normalized p-adic absolute value of f.

$\boxed{\text{Archimedian factors}}$: For $z \in \mathbb{C}$ and $a \in \mathbb{R}$, define

$$\mathcal{O}_k^{+,(z)}(a) = \frac{2\pi}{\Gamma(k)} \frac{\Gamma\left(k+\frac{z-1}{2}\right)\Gamma\left(k+\frac{-z-1}{2}\right)}{\Gamma_{\mathbb{R}}\left(\frac{1+z}{2}\right)\Gamma_{\mathbb{R}}\left(\frac{1-z}{2}\right)} \delta(|a|>1)\,(a^2-1)^{1/2}\mathfrak{P}^{1-k}_{\frac{z-1}{2}}(|a|),$$

$$\mathcal{O}_k^{-,(z)}(a) = \frac{\pi i}{\Gamma(k)}\Gamma\left(k+\tfrac{z-1}{2}\right)\Gamma\left(k+\tfrac{-z-1}{2}\right)\operatorname{sgn}(a)\,(1+a^2)^{1/2}\{\mathfrak{P}^{1-k}_{\frac{z-1}{2}}(ia)-\mathfrak{P}^{1-k}_{\frac{z-1}{2}}(-ia)\},$$

where $\mathfrak{P}^{\mu}_{\nu}(x)$ is the associated Legendre function of the 1st kind:

$$\mathfrak{P}^{\mu}_{\nu}(x) = \Gamma(1-\mu)^{-1}(x+1)^{\mu/2}(x-1)^{-\mu/2}{}_2F_1\left(-\nu,\nu+1;1-\mu;\tfrac{1-x}{2}\right).$$

Note that $\mathcal{O}_k^{-,(z)}(0)$ is understood as $\lim_{a\to 0}\mathcal{O}_k^{-,(z)}(a)$.

The normalized Eisenstein series is defined by

$$E^*(z,\tau) := \Lambda(z+1)E(z,\tau) \quad \text{with } \Lambda(s)=\Gamma_{\mathbb{R}}(s)\zeta(s).$$

The completed L-function of $E^*(z)$ is defined as

$$\Lambda(s,E^*(z)) = \Lambda\left(s+\tfrac{z}{2}\right)\,\Lambda\left(s+\tfrac{z}{2}\right).$$

Theorem 3.1. *Let $k \in 2\mathbb{N}$ with $k \geqslant 4$.*

(1) *For $n \in \mathbb{N}$, $D \in \mathscr{D}$, set*

$$\mathcal{T}(n,D) = \{t\in\mathbb{Z}|\, t^2-4n = f^2D\,(\exists f\in\mathbb{N})\,\}.$$

Let ϕ be an even Hecke-Maass cusp form, or the normalized Eisenstein series $E^*(z)$ with $|\operatorname{Re} z| < k-3$.

Let $(\nu_\infty, \nu_{\mathrm{fin}} = \{\nu_p\}_{p<\infty})$ be the parameter of Laplace-Hecke eigenvalues of ϕ that is defined by the relations

$$\lambda_\infty = \frac{1}{4}(1-\nu_\infty)(1+\nu_\infty) \quad \lambda_p = p^{\nu_p/2}+p^{-\nu_p/2}$$

from the eigenvalues of Laplacian and the Hecke operators. Then,

$$\frac{4\pi\sqrt{n}}{k-1}\sum_{h\in H_k(1)}\lambda_h(n)\,\mu_\phi(h) = \mathbb{J}_{\mathrm{u}}(\phi,n)+\mathbb{J}_{\mathrm{hyp}}(\phi,n)+\mathbb{J}_{\mathrm{ell}}(\phi,n), \tag{3.1}$$

where

$$\mathbb{J}_{\mathrm{hyp}}(\phi,n) = \frac{2^{\delta(\phi)}}{4}\Lambda\left(\frac{1}{2},\phi\right)\sum_{\substack{n=d_1d_2\\ d_1,d_2>0, d_1\neq d_2}}\mathbf{B}(\nu_{\mathrm{fin}};(d_1-d_2)^2)\mathcal{O}_k^{+,(\nu_\infty)}\left(\frac{d_1+d_2}{d_1-d_2}\right),$$

$$\mathbb{J}_{\mathrm{ell}}(\phi,n) = \frac{1}{2}\sum_{D\in\mathscr{D}}2^{\delta(D<0)}\,\mathbb{P}_D(\phi)\sum_{t\in\mathcal{T}(n,D)}\mathbf{B}(\nu_{\mathrm{fin}};t^2-4n)\,\mathcal{O}_k^{\operatorname{sgn}(t^2-4n),(\nu_\infty)}\left(\frac{t}{\sqrt{|t^2-4n|}}\right),$$

$\mathbb{J}_{\mathrm{u}}(\phi,n) = 0$ *and* $\delta(\phi)=0$ *if ϕ is a cusp form and*

$$\mathbb{J}_{\mathrm{u}}(\phi,n) = n^{1/2}\Lambda\left(\frac{z+1}{2}\right)\delta(n=\square)\left\{\Lambda(-z)\,2^{1-z}\pi^{(3-z)/4}\frac{\Gamma(k+(z-1)/2)}{\Gamma(k)\Gamma((z+1)/4)}\,n^{-(z+1)/4}\right.$$
$$\left.+\,\Lambda(z)\,2^{1+z}\pi^{(3+z)/4}\frac{\Gamma(k+(-z-1)/2)}{\Gamma(k)\Gamma((-z+1)/4)}\,n^{-(-z+1)/4}\right\}$$

and $\delta(\phi)=1$ if $\phi = E^*(z)$. In the last formula, $\delta(n=\square)$ is 1 if n is a perfect square and is 0 otherwise.

(2) *We have an analogous formula for holomorphic cusp forms h with square-free level N and an even Hecke-Maass form ϕ on $\mathrm{SL}_2(\mathbb{Z})$. More generally in the setting of adeles, we have a similar result for Hilbert modular forms of square-free level over an arbitrary totally real number field F such that the place 2 of $\mathbb{Q}$ completely splits in the extension F.*

For detail, we refer to [10] and [11]. The above formula for $\phi = E^*(z)$ has a different shape from [18, Theorem 1]. For an adjustment to see that the two formulas are indeed the same, we use the relations

$$\mu_h(E^*(z)) = \frac{\Lambda\left(\frac{z+1}{2}\right)\Lambda\left(\frac{z+1}{2}, h, \mathrm{sym}^2\right)}{2\,\Lambda(1, h, \mathrm{sym}^2)}, \quad \text{(Rankin-Selberg integral)},$$
$$\mathbb{P}_D(E^*(z)) = 2^{-\delta(D<0)}|D|^{(z+1)/4}\,\Lambda\left(\tfrac{z+1}{2}\right)\Lambda\left(\tfrac{z+1}{2}, \chi_D\right) \quad (D \in \mathscr{D}), \quad \text{(Hecke's formula)}$$

in conjunction with a more elementary formula

$$\sum_{0<d|f} \mu(d)\left(\tfrac{D}{d}\right) d^{-(z+1)/2}\sigma_{-z}(f/d) = f^{-(z+1)/2}\mathbf{B}(\underline{z}; \Delta),$$

with $\underline{z}$ the diagonal image of z in $\prod_p(\mathbb{C}/4\pi i(\log p)^{-1}\mathbb{Z})$ for any non-zero discriminant $\Delta = f^2 D$. Here $\mu(d)$ is the Möbius function and $\sigma_{-z}(n) := \sum_{0<d|n} d^{-z}$.

3.2. **Asymptotic formulas.** Let ϕ and $H_k(1)$ be as above. For $n \in \mathbb{N}$ and $k \in 2\mathbb{N}$, consider the average

$$\mathbf{A}_{k,n}(\phi) = \sum_{h \in H_k(1)} \lambda_h(n)\,\mu_h(\phi)$$

Theorem 3.2. *Let $\nu_{\mathrm{fin}} = \{\nu_p\}_{p<\infty}$ be the parameter of Hecke eigenvalues of ϕ. Then for any $n \in \mathbb{N}$, we have*

$$(-1)^{k/2}\sqrt{n}\mathbf{A}_{k,n}(\phi) = \tfrac{1}{4}\mathbb{P}_{-4n}(\phi)\mathbf{B}(\nu_{\mathrm{fin}}; -4n) + \sum_{\substack{t\in\mathbb{Z},\\ 0<|t|<2\sqrt{n}}} \mathbb{P}_{t^2-4n}(\phi)\mathbf{B}(\nu_{\mathrm{fin}}; t^2-4n)\sqrt{|\Delta|}^{-1}\left\{\rho\,(\bar\rho/\rho)^{k/2} + \bar\rho\,(\rho/\bar\rho)^{k/2}\right\} + O(k^{-1})$$

as $k \in 2\mathbb{N}$, $k \to \infty$. Here $\Delta = t^2 - 4n$ and $\rho = 2^{-1}(-t + i\sqrt{|\Delta|})$.

Note that the second term in the right-hand side of the asymptotic formula is $O(1)$ with an oscilatory behavior. By taking a further average over $k \in [\mathrm{K}, 2\mathrm{K})$, it is absorbed into the error.

Theorem 3.3. *Let $\nu_{\mathrm{fin}} = \{\nu_p\}_{p<\infty}$ be the parameter of Hecke eigenvalues of ϕ. For any $n \in \mathbb{N}$,*

$$\lim_{\mathrm{K}\to\infty} \frac{2}{\mathrm{K}} \sum_{k\in(2\mathbb{N})\cap[\mathrm{K},2\mathrm{K})} (-1)^{k/2}\mathbf{A}_{k,n}(\phi) = \frac{1}{\sqrt{4n}}\mathbb{P}_{-4n}(\phi)\,\mathbf{B}(\nu_{\mathrm{fin}}; -4n).$$

3.3. **Non-vanishing results.** Let ϕ be an even Hecke-Maass cusp form on $\mathrm{SL}_2(\mathbb{Z})$ with the spectral parameter $(\nu_\infty, \nu_{\mathrm{fin}} = \{\nu_p\}_{p<\infty})$. Let us define $X(\phi)$ to be the set of $n \in \mathbb{N}$ with the following properties.

(i) $-4n \in \mathscr{D}$.
(ii) $L(1/2, \phi \otimes \chi_{-4n}) \neq 0$.

Note that $\epsilon(1/2, \pi_\phi) = +1$. Then we have $\#X(\phi) = \infty$, which is a special case of [1, Theorem B1]. For our result to be non empty, we need the non-vanishing of the central value $L(1/2, \phi)$. Up to now, no single example of such ϕ has been ever constructed. However, abundance of such ϕ is known through asymptotic formulas. Let $\{\phi_j\}_{j=1}^{\infty}$ be a complete orthogormal system of the space of even Hecke-Maass cuspforms on $\mathrm{SL}_2(\mathbb{Z})$ such that $\Delta\phi_j = \frac{1-\nu_{j,\infty}^2}{4}\phi_j$ and such that in the corresponding set of eigenvalues $\lambda_j = (1-\nu_{j,\infty}^2)/4$ $(j \geqslant 1)$ each eigenvalue is counted with its multiplicity. Here we quote the following asymptotic formula due to Motohashi [16, Theorem 2].

$$\sum_{|\nu_{j,\infty}/2| \leqslant t} \frac{L(1/2, \phi_j)}{\cos(\pi\nu_{j,\infty}/2)} \sim \frac{2}{\pi^2} t^2 \log t \quad (t \to +\infty).$$

This evidently shows that there are infinitely many ϕ with $L(1/2, \phi) \neq 0$.

Let ϕ be an even Hecke-Maass cusp form on $\mathrm{SL}_2(\mathbb{Z})$. For $n \in \mathbb{N}$ and $\mathrm{K} > 0$, define

$$N_{\phi,n}(\mathrm{K}) = \# \left\{ h \in \bigcup_{k \in 2\mathbb{N} \cap [\mathrm{K}, 2\mathrm{K})} H_k(1) \;\middle|\; L\left(\tfrac{1}{2}, \phi \times \mathrm{Ad}\, h\right) \neq 0,\ \lambda_h(n) \neq 0 \right\}.$$

Theorem 3.4. *Let ϕ be an even Hecke-Maass form on $\mathrm{SL}_2(\mathbb{Z})$ with the Laplace eigenvalue $(1-\nu_\infty^2)/4$. We suppose*

$$L\left(\tfrac{1}{2}, \phi\right) \neq 0.$$

For any $n \in X(\phi)$ and $\varepsilon \in (0,1)$, there exists a constant $\mathrm{K}(\phi, n, \varepsilon) > 0$ such that

$$\frac{N_{\phi,n}(\mathrm{K})}{\mathrm{K}} \geqslant \left(\frac{1-\varepsilon}{16\pi}\right) \frac{1}{n\, d(n)^2} \frac{L(1/2, \phi \otimes \chi_{-4n})}{C(\phi)\, L(1, \phi, \mathrm{sym}^2)}, \tag{3.2}$$

for all $\mathrm{K} > \mathrm{K}(\phi, n, \varepsilon)$, where $d(n)$ is the number of positive divisors of n.

Note that the right-hand side of (3.2) is positive for $n \in X(\phi)$. This theorem shows that among the family of L-functions

$$L(s, \phi \times \mathrm{Ad}\, h), \quad (h \in \bigcup_{k \in 2\mathbb{N} \cap [\mathrm{K}, 2\mathrm{K})} H_k(1))$$

the number of non-vanishing central value is at least a positive multiple of K asymptotically as $\mathrm{K} \to \infty$, whereas the size of the whole family is about K^2. Note that the infinitum of non-vanishing central values among the family is a direct consequence of the variance formula (1.7); Theorem 3.4 is viewed as a quantification of this fact.

3.4. **Equidistribution theorem.** Let $N \in \mathbb{N}$ and $k \in 2\mathbb{N}$. Let q be a prime number such that $(N, q) = 1$. Consider the discrete measure on $[-2, 2]$ defined as

$$\mu_{N,k,q}, \varphi\rangle := \frac{1}{[\Gamma_0(1) : \Gamma_0(N)]} \sum_{h \in H_k(N)} \varphi\left(\lambda_h(q)\right), \quad \varphi \in C_{\mathrm{comp}}([-2, 2]).$$

The limiting behavior of the measure $\mu_{N,k,q}$ as $N + k \to \infty$ is determined in [8].

$$\frac{12}{k-1} \langle \mu_{N,k,q}, \varphi\rangle \longrightarrow \int_{-2}^{2} \varphi(x) \frac{1+q}{\pi} \frac{\sqrt{1-x^2/4}}{(q^{1/2}+q^{-1/2})^2 - x^2}\, \mathrm{d}x$$

as $k+N\to\infty$. For a square free level N, we define a discrete measure $\mu_{N,k,q}^{(z)}$ ($z\in[0,1]$) on $[-2,2]$ as

$$\langle \mu_{N,k,q}^{(z)},\varphi\rangle = \prod_{p|N}\frac{p^{(z-1)/2}}{1+p^{(z+1)/2}}\sum_{h\in H_k(N)} W_N^{(z)}(h)\frac{\Lambda(\frac{z+1}{2},h,\mathrm{sym}^2)}{\Lambda(1,h,\mathrm{sym}^2)}\,\varphi\left(\lambda_h(q)\right),$$

with

$$W^{(z)}(h)=(NN_h^{-1})^{(1-z)/2}\prod_{p|NN_h^{-1}}\left\{1+\frac{(p^{1/2}+p^{-1/2})(p^{z/2}+p^{-1/2})-\lambda_h(p)^2}{(p^{1/2}+p^{-1/2})^2-\lambda_h(p)^2}\right\},$$

where N_h denotes the conductor of h. Note that $\mu_{N,k,q}^{(1)}=\mu_{N,k,q}$. Then we have an analogous limit theorem of $\mu_{N,k,q}^{(z)}$ ($z\in[0,1]$) as $N\to\infty$ (with a fixed $k\geqslant 4$), where N is square free.

Theorem 3.5. *Suppose $k>4$ for simplicity. For any polynomial function $\varphi(x)$ on $[-2,2]$, we have*

$$\lim_{N\to\infty}\langle\mu_{N,k,q}^{(z)},\varphi\rangle = r(z)C_k(z)\int_{-2}^{2}\varphi(x)\frac{1+q^{(z+1)/2}}{\pi}\frac{\sqrt{1-x^2/4}}{(q^{(1+z)/4}+q^{-(1+z)/4})^2-x^2}\,\mathrm{d}x$$

uniformly in $z\in[0,1]$, where

$$r(z)=\begin{cases}\zeta(z+1) & (z>0),\\ 1 & (z=0)\end{cases},\qquad C_k(z)=2^{(3-z)/2}\pi^{-(3z+1)/4}\Gamma\left(\tfrac{z+3}{4}\right)\frac{\Gamma\left(k+\frac{z-1}{2}\right)}{4\pi\Gamma(k-1)}.$$

If the non-negativity $L(s,h,\mathrm{sym}^2)\geqslant 0$ ($\forall h\in\bigcup_N H_k(N)$, $\forall s\in[0,1]$) (k: fixed) were true, the limit formula holds for all $\varphi\in C_{\mathrm{comp}}([-2,2])$.

Theorem 3.6. *Suppose $k\in 2\mathbb{N}$ with $k>4$. Let q be a prime number.*

(1) *There exists $M>q$ such that, for any prime number $N>M$ and for any $z\in[0,1]$ there exists $h\in S_k^{\mathrm{new}}(\Gamma_0(N))$ with $L\left(\frac{z+1}{2},h,\mathrm{sym}^2\right)\neq 0$.*

(2) *Suppose the non-negativity : $L(s,h;\mathrm{sym}^2)\geqslant 0$ ($\forall\pi\,\forall s\in[0,1]$). Then given a subinterval $[\alpha,\beta]\subset[-2,2]$, there exists $M>q$ such that, for any prime number $N>M$ and for any $z\in[0,1]$ there exists $h\in S_k^{\mathrm{new}}(\Gamma_0(N))$ with the properties:*

$$L\left(\tfrac{z+1}{2},h,\mathrm{sym}^2\right)\neq 0,\quad \lambda_h(q)\in[\alpha,\beta].$$

4. A sketch of the proofs of Theorem 3.1

In this section, we give a brief idea of proof of Theorem 3.1. Most of the part, no accuracy is intended below. Set $G=\mathrm{PGL}(2)$ and $\mathbb{A}$ the adele ring of $\mathbb{Q}$. One considers the kernel function

$$K(g,g_1)=\sum_{\lambda\in G(\mathbb{Q})}\Phi(g^{-1}\lambda g_1),\quad g,g_1\in G(\mathbb{A})$$

where $\Phi=\Phi_\infty\otimes(\otimes_p\Phi_p)$ on $G(\mathbb{A})$ with Φ_∞ being the matrix coefficient of a discrete seris representation of $G(\mathbb{R})$ of weight k and Φ_p ($p<\infty$) being the characteristic function of $T(n)=\{g\in\mathrm{M}_2(\mathbb{Z}_p)|\det g\in n\mathbb{Z}_p^\times\}$. Then one computes the integral

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})}\phi(g)\,K(g,g)\,\mathrm{d}g \tag{4.1}$$

in two ways. The spectral expansion of the kernel function takes the form

$$K(g,g_1)=C\sum_{h\in H_k(1)}\lambda_h(n)\,\tilde{h}(g)\,\overline{\tilde{h}(g_1)},$$

where $\tilde{h}$ is a function on the adeles $G(\mathbb{A})$ corresponding to h. Plugging this formula to (4.1), we easily obtain an expression which is essentially yields the left-hand side of (3.1). To obtain the other side of (3.1), Jacquet-Zagier's approach uses the series expression of the Eisenstein series $\phi = E^*(z)$ on the convergence region $\operatorname{Re} z > 1$ and resorts to an unfolding argument. The same argument does not carry over to the case when ϕ is a cusp form. In our approach (as in [13]), we use the expression

$$K(g,g) = \sum_{[\gamma]} \sum_{\lambda \in G(\mathbb{Q})_\gamma \backslash G(\mathbb{Q})} \Phi(g^{-1}\lambda^{-1}\gamma\lambda g) = \sum_{[\gamma]} \mathscr{K}_\gamma(g),$$

where $[\gamma]$ runs through conjugacy classes of $G(\mathbb{Q})$, to have

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \phi(g)\, K(g,g)\, \mathrm{d}g = \sum_{[\gamma]} \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \phi(g)\, \mathscr{K}_\gamma(g)\, \mathrm{d}g.$$

Suppose ϕ is our cusp form (lifted to a function on $G(\mathbb{A})$). Then

$$\begin{aligned}
\int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \phi(g)\, \mathscr{K}_\gamma(g)\, \mathrm{d}g &= \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \phi(g) \left\{ \sum_{\lambda \in G(\mathbb{Q})_\gamma \backslash G(\mathbb{Q})} \Phi(g^{-1}\lambda^{-1}\gamma\lambda g) \right\} \mathrm{d}g \\
&= \int_{G(\mathbb{Q})_\gamma \backslash G(\mathbb{A})} \phi(g)\, \Phi(g^{-1}\gamma g)\, \mathrm{d}g \\
&= \int_{G(\mathbb{A})_\gamma \backslash G(\mathbb{A})} \underbrace{\left\{ \int_{G(\mathbb{Q})_\gamma \backslash G(\mathbb{A})_\gamma} \phi(\tau g) \mathrm{d}\tau \right\}}_{\mathscr{P}_\gamma(\phi : g)} \times \Phi(g^{-1}\gamma g)\, \mathrm{d}g.
\end{aligned}$$

If γ is trivial or the unipotent class, $\mathscr{P}_\gamma(\phi : g) = 0$ by the cuspidality of ϕ. Otherwise, $G_\gamma \cong E^\times$ with $E/\mathbb{Q}$ being a quadratic étale algebra (i.e., a quadratic field or $\mathbb{Q} \times \mathbb{Q}$). Note that

$$\phi = \otimes_v \phi_v \in \bigotimes_v \operatorname{Ind}_{B(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)} (|\,|_{\mathbb{A}}^{\nu_v/2} \boxtimes |\,|_{\mathbb{A}}^{-\nu_v/2})^{K_v}$$

where $K_p = \mathrm{GL}_2(\mathbb{Z}_p)$ and $K_\infty = O(2)$. By uniqueness of the toric model (due to Waldspurger)

$$\dim_{\mathbb{C}} \operatorname{Hom}_{G(\mathbb{Q}_v)} \left(\operatorname{Ind}_{B(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)} (|\,|_v^{\nu/2} \boxtimes |\,|_v^{-\nu/2}), C^\infty(E_v^\times \backslash G(\mathbb{Q}_v)) \right) = 1,$$

we have a decomposition

$$\mathscr{P}_\gamma(\phi : g) = \mathscr{P}_\gamma(\phi) \prod_v \Xi_v^E(\nu_v; g_v), \quad g = (g_v) \in G(\mathbb{A}),$$

where $\Xi_v^E(\nu; g_v)$ is the unramified spherical function on $E_v^\times \backslash G(\mathbb{Q}_v)$ whose explicit formula is easily worked out. (A novelty in our method lies in the usage of these materials from representation theory.) In this way, we have the Euler product expression:

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \phi(g)\, \mathscr{K}_\gamma(g)\, \mathrm{d}g = \mathscr{P}_\gamma(\phi) \prod_v \int_{G(\mathbb{Q}_v)_\gamma \backslash G(\mathbb{Q}_v)} \Xi_v^E(\nu_v : g_v)\, \Phi_v(g_v)\, \mathrm{d}g_v.$$

with a constant $\mathscr{P}_\gamma(\phi)$ which is eventually identified with $\mathbb{P}_D(\phi)$ up to a tractable constant if D is the discriminant of the quadratic field $\mathbb{Q}[\gamma]$. A brute-force computation gives us an explicit formula of each local integral.

Let $\phi = E^*(z)$, which is not of rapid decay on a fundamental domain. To remedy this, we consider the smoothed Eisensteis series

$$\phi_\beta(g) = \int_{(c)} E^*(z,g)\,\beta(z)\,\mathrm{d}z,$$

with $\beta(z)$ an entire function such that $\beta(z) = O((1+|\mathrm{Im} z|)^{-N})$, $\beta(\pm 1) = \beta'(\pm 1) = 0$. A similar analysis goes through as it is before, except that the unipotent term does not vanish this time. $\square$

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